LI ALGEBRALARINING LOKAL VA 2-LOKAL ANTI-DIFFERENSIALLASHLARI

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ANNOTATSIYA

Ushbu tezisda chekli o'lchamli Li algebralarining lokal va 2-lokal anti-differensiallashlari o'rganiladi. Anti-differensiallash deb $\varphi([x, y]) = -[\varphi(x), y] - [x, \varphi(y)]$ shartni qanoatlantiruvchi chiziqli akslantirishga aytiladi. Xarakteristikasi nolga teng algebraik yopiq maydon ustida sl_2 algebrasining barcha anti-differensiallashlari tavsiflanadi va ular besh o'lchamli fazo tashkil etishi ko'rsatiladi. sl_2 ning ixtiyoriy chiziqli akslantirishi lokal anti-differensiallash ekanligi, demak, sl_2 anti-differensiallash bo'lmagan lokal anti-differensiallashlarga ega ekanligi isbotlanadi. Shu bilan birga yarim oddiy Li algebralarining har bir 2-lokal anti-differensiallashi anti-differensiallash ekanligi, lokal holda esa bu xossa faqat sl_2 ga izomorf idealga ega bo'lmagan algebralar uchun o'rinli ekanligi ko'rsatiladi.

Kalit so'zlar: Li algebrasi, anti-differensiallash, δ-differensiallash, lokal anti-differensiallash, 2-lokal anti-differensiallash, oddiy Li algebrasi, yarim oddiy Li algebrasi, sl_2.

ЛОКАЛЬНЫЕ И 2-ЛОКАЛЬНЫЕ АНТИДИФФЕРЕНЦИРОВАНИЯ АЛГЕБР ЛИ

АННОТАЦИЯ

В тезисе изучаются локальные и 2-локальные антидифференцирования конечномерных алгебр Ли. Антидифференцированием называется линейное отображение, удовлетворяющее условию $\varphi([x, y]) = -[\varphi(x), y] - [x, \varphi(y)]$. Над алгебраически замкнутым полем нулевой характеристики описаны все антидифференцирования алгебры sl_2 и показано, что они образуют пятимерное пространство. Доказано, что любое линейное отображение sl_2 является локальным антидифференцированием; следовательно, sl_2 обладает локальными антидифференцированиями, не являющимися антидифференцированиями. Вместе с тем показано, что каждое 2-локальное антидифференцирование полупростой алгебры Ли является антидифференцированием, а в локальном случае это свойство выполняется только для алгебр, не содержащих идеалов, изоморфных sl_2.

Ключевые слова: алгебра Ли, антидифференцирование, δ-дифференцирование, локальное антидифференцирование, 2-локальное антидифференцирование, простая алгебра Ли, полупростая алгебра Ли, sl_2.

LOCAL AND 2-LOCAL ANTI-DERIVATIONS OF LIE ALGEBRAS

ABSTRACT

This thesis studies local and 2-local anti-derivations of finite-dimensional Lie algebras. An anti-derivation is a linear map satisfying $\varphi([x, y]) = -[\varphi(x), y] - [x, \varphi(y)]$. Over an algebraically closed field of characteristic zero, all anti-derivations of sl_2 are described, and they are shown to form a five-dimensional space. We prove that every linear map on sl_2 is a local anti-derivation; hence sl_2 admits local anti-derivations which are not anti-derivations. At the same time, every 2-local anti-derivation of a semisimple Lie algebra is shown to be an anti-derivation, while in the local case this property holds only for algebras having no ideals isomorphic to sl_2.

Keywords: Lie algebra, anti-derivation, δ-derivation, local anti-derivation, 2-local anti-derivation, simple Lie algebra, semisimple Lie algebra, sl_2.

1. KIRISH

Algebralarning lokal akslantirishlari nazariyasi R. Kadison [2] hamda D. Larson va A. Sourour [3] ishlaridan boshlangan: ular operator algebralarida har bir lokal differensiallash haqiqiy differensiallash bo‘lishini ko‘rsatdilar. 1997-yilda P. Šemrl [1] 2-lokal differensiallash va 2-lokal avtomorfizm tushunchalarini kiritdi. Li algebralari uchun Sh.A. Ayupov va K.K. Kudaybergenov [4] chekli o‘lchamli yarim oddiy Li algebralarining har bir lokal differensiallashi differensiallash ekanligini isbotladilar, nilpotent Li algebralarida esa differensiallash bo‘lmagan lokal differensiallashlar mavjudligini ko‘rsatdilar; 2-lokal differensiallashlar uchun shunga o‘xshash natija [5] ishda olingan.

Differensiallash tushunchasining tabiiy umumlashmasi δ-differensiallashdir: $\varphi([x, y]) = \delta([\varphi(x), y] + [x, \varphi(y)])$. $\delta = -1$ bo‘lganda anti-differensiallash hosil bo‘ladi; bu tushuncha N. Jacobson kitobida [6] uchraydi. V.T. Filippov [7, 8] δ-differensiallashlarni sistemali o‘rganib, tub Li algebralari $\delta \neq -1, 0, 1/2, 1$ bo‘lganda nolmas δ-differensiallashlarga ega emasligini hamda nolmas anti-differensiallashga ega markaziy oddiy Li algebrasi sl_2 ning formasi ekanligini isbotladi. So‘nggi yillarda lokal va 2-lokal δ-differensiallashlar, jumladan, 1/2-differensiallashlar [9] va assotsiativ algebralarning anti-differensiallashlari [10] faol o‘rganilmoqda; Li algebralari uchun lokal va 2-lokal anti-differensiallashlar [11] ishda qaralgan.

Ushbu tezisning maqsadi — oddiy va yarim oddiy Li algebralari, xususan, sl_2 algebrasi uchun lokal va 2-lokal anti-differensiallashlar haqiqiy anti-differensiallash bo‘ladimi, degan savolga to‘liq javob berishdir.

2. ASOSIY TUSHUNCHALAR

Bundan buyon F — xarakteristikasi nolga teng algebraik yopiq maydon, L — F ustidagi chekli o‘lchamli Li algebrasi.

Ta’rif 1. Chiziqli $\varphi: L \to L$ akslantirish **anti-differensiallash** deyiladi, agar barcha $x, y \in L$ uchun

$$\varphi([x, y]) = -[\varphi(x), y] - [x, \varphi(y)] \qquad (1)$$

tenglik bajarilsa. Barcha anti-differensiallashlar fazosini AntiDer(L) bilan belgilaymiz.

Ta’rif 2. Chiziqli $\varphi: L \to L$ akslantirish **lokal anti-differensiallash** deyiladi, agar har bir $x \in L$ uchun shunday $D_x \in \text{AntiDer}(L)$ mavjud bo‘lsaki, $\varphi(x) = D_x(x)$ bo‘lsa.

Ta’rif 3. $\varphi: L \to L$ akslantirish (chiziqli bo‘lishi talab qilinmaydi) **2-lokal anti-differensiallash** deyiladi, agar ixtiyoriy $x, y \in L$ uchun shunday $D_{x,y} \in \text{AntiDer}(L)$ topilsaki, $\varphi(x) = D_{x,y}(x)$ va $\varphi(y) = D_{x,y}(y)$ bo‘lsa.

Ravshanki, har bir anti-differensiallash ham lokal, ham 2-lokal anti-differensiallashdir. Asosiy masala — teskari tasdiqning o‘rinli yoki o‘rinli emasligini aniqlash.

Izoh. Differensiallashlardan farqli ravishda, ichki akslantirish $ad_a(x) = [a, x]$ umuman olganda anti-differensiallash bo‘lmaydi. Haqiqatan, agar ad_a bir vaqtda differensiallash va anti-differensiallash bo‘lsa, (1) dan $ad_a([x, y]) = 0$ kelib chiqadi; $L = [L, L]$ bo‘lgan oddiy algebrada bu $a \in Z(L) = \{0\}$ ni beradi.

3. ASOSIY NATIJALAR

$sl_2 = \text{span}\{e, f, h\}$ algebrasini standart $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$ ko‘paytmalar bilan qaraymiz.

Teorema 1. Chiziqli $\varphi: sl_2 \to sl_2$ akslantirish anti-differensiallash bo‘lishi uchun shunday $\alpha, \beta, \gamma, \delta, \varepsilon \in F$ mavjud bo‘lishi zarur va yetarliki,

$$\varphi(e) = -\alpha e + \beta f + \gamma h, \quad \varphi(f) = \delta e - \alpha f + \varepsilon h, \quad \varphi(h) = 2\varepsilon e + 2\gamma f + 2\alpha h \qquad (2)$$

bo‘lsa. Xususan, $\dim \text{AntiDer}(sl_2) = 5$.

Isbot. (1) shartni bazis elementlarning (h, e), (h, f), (e, f) juftliklari uchun yozib, φ matritsasining elementlariga nisbatan chiziqli bir jinsli tenglamalar sistemasini hosil qilamiz. Uni yechish (2) ko‘rinishni beradi. Masalan, $[e, f] = h$ uchun $\varphi(h) = -[\varphi(e), f] - [e, \varphi(f)] = 2\varepsilon e + 2\gamma f + 2\alpha h$. Teskarisi bevosita tekshiriladi. □

Teorema 2. sl_2 ning ixtiyoriy chiziqli akslantirishi lokal anti-differensiallashdir. Xususan, sl_2 da anti-differensiallash bo‘lmagan lokal anti-differensiallashlar mavjud.

Isbot. $x = ae + bf + ch \neq 0$ bo‘lsin. (2) ga ko‘ra ixtiyoriy $D \in \text{AntiDer}(sl_2)$ uchun

$$D(x) = (-a\alpha + b\delta + 2c\varepsilon)e + (a\beta - b\alpha + 2c\gamma)f + (a\gamma + b\varepsilon + 2c\alpha)h \qquad (3)$$

$a \neq 0$; $a = 0$, $b \neq 0$; $a = b = 0$, $c \neq 0$ hollarini alohida qarab, $(\alpha, \beta, \gamma, \delta, \varepsilon) \mapsto D(x)$ akslantirish sl_2 ga syurektiv ekanligini ko‘rsatamiz. Demak, har bir chiziqli φ va har bir x uchun $\varphi(x) = D_x(x)$ bo‘ladigan $D_x \in AntiDer(sl_2)$ mavjud. Xususan, ayniy akslantirish id lokal anti-differensiallash, lekin $id([x, y]) = [x, y] \neq -2[x, y]$ bo‘lgani uchun anti-differensiallash emas. □

Teorema 3. sl_2 ning har bir 2-lokal anti-differensiallashi anti-differensiallashdir.

Isbot. φ — 2-lokal anti-differensiallash bo‘lsin. $\psi = \varphi - D_{h,h}$ ham 2-lokal anti-differensiallash va $\psi(h) = 0$. (2) dan $D(h) = 0$ shart $\alpha = \gamma = \varepsilon = 0$ ni beradi, shuning uchun (h, y) juftlik bo‘yicha $y = ae + bf + ch$ uchun $\psi(y) = b\delta_y e + a\beta_y f$. $D_1(e) = \beta_e f$, $D_1(f) = \delta_f e$, $D_1(h) = 0$ anti-differensiallashni olib, $\chi = \psi - D_1$ ni qaraymiz: $\chi(e) = \chi(f) = \chi(h) = 0$ va $\chi(y)$ ham yuqoridagi ko‘rinishga ega. (e, y) juftlik uchun $D(e) = 0$ shartdan $\alpha = \beta = \gamma = 0$, demak $\chi(y)$ ning f-komponentasi nolga teng; (f, y) juftlik uchun $D(f) = 0$ shartdan $\alpha = \delta = \varepsilon = 0$, demak e-komponentasi ham nolga teng. Shunday qilib, $\chi \equiv 0$ va $\varphi = D_{h,h} + D_1 \in AntiDer(sl_2)$. □

Lemma 1. $L = L_1 \oplus \ldots \oplus L_k$ yarim oddiy Li algebrasi bo‘lsa, $AntiDer(L) = AntiDer(L_1) \oplus \ldots \oplus AntiDer(L_k)$.

Isbot. $x \in L_i$, $y \in L_j$, $j \neq i$ bo‘lsin. $[x, y] = 0$ bo‘lgani uchun (1) dan $[\varphi(x), y] + [x, \varphi(y)] = 0$. Bu tenglikni L_j ga proyeksiyalasak, $[\varphi(x)_j, y] = 0$ barcha $y \in L_j$ uchun, ya’ni $\varphi(x)_j \in Z(L_j) = \{0\}$. Demak, $\varphi(L_i) \subseteq L_i$. □

Teorema 4. L — yarim oddiy Li algebrasi bo‘lsin. U holda:

a) L ning har bir 2-lokal anti-differensiallashi anti-differensiallashdir;

b) L ning har bir lokal anti-differensiallashi anti-differensiallash bo‘lishi uchun L ning sl_2 ga izomorf idealga ega bo‘lmasligi zarur va yetarli; bu holda $AntiDer(L) = \{0\}$.

Isbot. a) $x = x_1 + \ldots + x_k$, $x_i \in L_i$ bo‘lsin. $y = x_i$ deb olsak, Lemma 1 ga ko‘ra $D_{x,y}$ ning L_i-komponentasi x_i da $\varphi(x_i)$ ga teng, shuning uchun $\varphi(x) = \varphi(x_1) + \ldots + \varphi(x_k)$ va $\varphi|L_i$ — L_i ning 2-lokal anti-differensiallashi. Filippov teoremasiga [8] ko‘ra $L_i \ncong sl_2$ bo‘lsa, $AntiDer(L_i) = \{0\}$ va $\varphi|L_i = 0$; $L_i \cong sl_2$ bo‘lsa, Teorema 3 qo‘llaniladi. b) Agar sl_2 ga izomorf ideal bo‘lmasa, $AntiDer(L) = \{0\}$, shuning uchun $\varphi(x) = D_x(x) = 0$. Aks holda $L_1 \cong sl_2$ bo‘lsin; L_1 da ayniy, qolgan komponentalarda nol bo‘lgan akslantirish Teorema 2 ga ko‘ra lokal anti-differensiallash, lekin anti-differensiallash emas. □

4. XULOSA

Olingan natijalar anti-differensiallashlar differensiallashlardan keskin farq qilishini ko‘rsatadi. Yarim oddiy Li algebralarida har bir lokal differensiallash differensiallash bo‘lsa-da [4], sl_2 algebrasida lokal anti-differensiallashlar fazosi barcha chiziqli akslantirishlar fazosi bilan ustma-ust tushadi. 2-lokal holda esa “ijobiy” natija saqlanib qoladi: yarim oddiy Li algebralarining har bir 2-lokal anti-differensiallashi anti-differensiallashdir. Keyingi tadqiqotlar uchun yechiluvchan va nilpotent Li algebralari, Leybnits algebralari hamda Li superalgebralarining lokal va 2-lokal anti-differensiallashlarini o‘rganish dolzarb hisoblanadi.

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