



GIBBS MEASURES OF THE ISING MODEL ON A CAYLEY TREE

Turdaliyeva Dildora Roziqjon qizi

Abstract: This thesis examines the Gibbs measures of the Ising model on a Cayley tree, focusing on their mathematical properties, existence, uniqueness, and relationship with phase transitions. The Ising model is one of the fundamental models in statistical mechanics and probability theory. It describes systems consisting of interacting spins that can take one of two possible values. The Cayley tree provides a useful mathematical structure for investigating these systems because its branching geometry allows the analysis of spin interactions through recursive equations. Particular attention is given to the Hamiltonian of the Ising model, the definition of Gibbs measures, boundary conditions, and the influence of temperature on equilibrium states. The thesis also discusses the role of critical temperature in the appearance of multiple translation-invariant Gibbs measures. The study demonstrates that the structure of the Cayley tree and the choice of boundary conditions are essential for understanding the equilibrium behavior of the Ising model. These mathematical results contribute to the broader study of statistical mechanics, probabilistic models, and complex interacting systems.

Keywords: Ising model, Cayley tree, Gibbs measures, statistical mechanics, probability theory, spin systems, boundary conditions, phase transition, critical temperature, Hamiltonian, equilibrium states, recursive equations, magnetization, thermodynamic limit.

The Ising model is one of the most widely studied mathematical models in statistical mechanics. It was originally developed to investigate the magnetic properties of materials and has subsequently become an important framework for understanding interacting particle systems. Despite its relatively simple definition, the model exhibits complex collective behavior, including spontaneous magnetization, multiple equilibrium states, and phase transitions.



These properties make it a significant subject of research in mathematical physics and probability theory. The basic idea of the Ising model is to represent a physical system through a collection of sites, each associated with a spin variable. Every spin can take one of two values, conventionally denoted by $+1$ and -1 . The energy of the entire system depends on the interactions between neighboring spins and, when present, an external magnetic field. The statistical behavior of the system is described by probability distributions that assign different probabilities to possible spin configurations [1].

One of the most important concepts in this context is the Gibbs measure. A Gibbs measure describes the equilibrium probability distribution of an interacting system at a given temperature. It provides a mathematical framework for studying the probability of different configurations and for calculating important physical quantities, such as magnetization and correlation functions. The existence and uniqueness of Gibbs measures are central questions in the mathematical theory of statistical mechanics. The geometry of the underlying structure has a substantial influence on the properties of an Ising model. Models can be defined on regular lattices, finite graphs, infinite graphs, and tree-like structures. Among these structures, the Cayley tree is particularly useful because it has a regular branching structure and contains no cycles. These properties allow many calculations to be expressed through recursive relations. The study of Gibbs measures on a Cayley tree is important because the boundary of a finite tree can have a considerable influence on the behavior of the entire system. Unlike many familiar lattice structures, a substantial proportion of the vertices of a large finite tree may lie near its boundary. Consequently, boundary conditions can affect the limiting probability distributions and the possible equilibrium states of the infinite system [2].

The main purpose of this thesis is to explain the mathematical foundations of the Ising model on a Cayley tree and to investigate the principal properties of



its Gibbs measures. The discussion focuses on the Hamiltonian, probability distributions, recursive equations, phase transitions, and the conditions under which multiple equilibrium measures may arise. A Cayley tree is a connected graph without cycles in which the vertices are arranged in a branching structure. In a regular Cayley tree, the number of neighboring vertices is determined by a fixed branching parameter. When the tree is viewed from a distinguished root, its vertices can be divided into successive generations or levels [3].

The root represents the first level, and its descendants form the subsequent levels. Each vertex, except the root, has one parent and a specified number of children. This regular structure provides a natural way to organize calculations involving interactions between neighboring vertices.

Let (k) denote the number of children of each vertex. In this convention, the root has (k) children, while every other vertex has one parent and (k) children. The total number of neighbors of a non-root vertex is therefore $(k+1)$. The absence of cycles is one of the most important features of the Cayley tree. In a graph containing cycles, interactions can create complicated dependencies between different parts of the system. On a tree, however, the unique path between any two vertices simplifies the analysis of these dependencies. This makes it possible to express the behavior of a vertex in terms of the statistical information associated with its neighboring branches. The branching structure also allows the construction of finite subtrees that gradually increase in size. By studying probability distributions on these finite subtrees and examining their limiting behavior, researchers can investigate Gibbs measures on the infinite tree [4].

Nevertheless, the Cayley tree has an important characteristic that must be considered carefully. As the number of levels increases, the boundary may contain a significant proportion of all vertices. Consequently, the choice of boundary conditions can influence the limiting behavior of the model. This





feature distinguishes the Cayley tree from several other infinite graph structures and makes boundary analysis especially important. This expression applies to the stated convention for the branching parameter and identifies the threshold for the appearance of nonzero solutions in the translation-invariant recursion. It should not be interpreted as a universal criterion for the uniqueness of every Gibbs measure under all possible boundary conditions. Above the corresponding critical temperature, thermal fluctuations dominate more strongly, and the zero solution is locally stable in the scalar recursion. Below the critical temperature, the recursion can admit additional nonzero solutions, indicating the possibility of ordered states.

The relationship between recursive solutions and physical phases requires careful interpretation. A solution of the recursion defines a candidate boundary law, and under the appropriate consistency conditions it determines a Gibbs measure. Nevertheless, a complete classification of Gibbs measures may require methods that go beyond the analysis of translation-invariant solutions. The role of boundary conditions is particularly significant on a Cayley tree. For example, fixing all boundary spins to $+1$ can produce different finite-volume distributions from those obtained by fixing all boundary spins to -1 . The limits of these distributions may correspond to different Gibbs measures, depending on the parameters of the model. This sensitivity to boundary conditions is one of the most interesting characteristics of the Ising model on a tree. It demonstrates that the geometry of the underlying graph can strongly influence the relationship between local interactions and global equilibrium behavior.

Conclusion

The study of Gibbs measures of the Ising model on a Cayley tree provides important insights into the mathematical foundations of statistical mechanics and probability theory. The model describes a system of interacting spins whose equilibrium behavior depends on the interaction strength, temperature, external



magnetic field, and boundary conditions. The Cayley tree is particularly suitable for this investigation because its branching structure allows the derivation of recursive equations. These equations help explain how local interactions influence the statistical behavior of the entire system and provide a framework for analyzing translation-invariant equilibrium states. Gibbs measures play a central role in this analysis because they describe the probability distributions associated with equilibrium configurations. Their existence and uniqueness are fundamental mathematical questions, while the possibility of multiple Gibbs measures is closely related to the structure of equilibrium phases. The analysis of recursive equations also provides a way to identify critical conditions under which additional translation-invariant solutions may emerge. Such solutions can be associated with ordered states and phase transitions, although the full classification of Gibbs measures requires consideration of more general boundary conditions and possible non-translation-invariant structures.

REFERENCES

1. Baxter, R. J. Exactly Solved Models in Statistical Mechanics. Academic Press, 2015
2. Georgii, H.-O. Gibbs Measures and Phase Transitions. Second Edition. De Gruyter, 2011.
3. Dobrushin, R. L. "The Description of a Random Field by Means of Conditional Probabilities and Conditions of Its Regularity." Theory of Probability and Its Applications, 2019.
4. d, O. E., and Ruelle, D. "Observables at Infinity and States with Short Range Correlations in Statistical Mechanics." Communications in Mathematical Physics, 2018
5. Rozikov, U. A. Gibbs Measures on Cayley Trees. World Scientific, 2013.



6. Bleher, P. M., and Ganikhodjaev, N. N. "On Pure Phases of the Ising Model on the Bethe Lattice." Theory of Probability and Its Applications, 2018.